CTP #1213
Fermilab Pub 84-108-T
October, 1984

On the Energy Crisis in Anti de Sitter Supersymmetry

William A. Bardeen
Fermilab, P.O. Box 500
Batavia, IL 60510

and

Daniel Z. Freedman*
Department of Mathematics and
Center for Theoretical Physics, Laboratory for Nuclear Science
Massachusetts Institute of Technology
Cambridge, MA 02139

Abstract

The energy operator in AdS supersymmetry is formally positive, i.e., $H \sim \sum_\alpha \{Q_\alpha, Q_\alpha^+\}$, but in supersymmetric field theories the standard energy density is not necessarily positive. Further, calculations of $<H>$ by ζ-function regularization of the sum of boson and fermion mode energies in a supermultiplet gives non-zero results. In two-dimensional interacting field theories with AdS supersymmetry, we calculate the energy operator and the other O(2,1) generators from the supercharge anticommutator $\{Q_\alpha, \overline{Q}_\beta\}$ by canonical methods, and we find generators which differ from the standard ones by surface terms. The improved energy density is non-negative and vanishes only in supersymmetric configurations. We prove that radiative corrections to the improved $<H>$ vanish to all orders in perturbation theory in supersymmetric vacua.

*Supported in part by NSF Grant No. 8407109-PHY and Department of Energy Contract DE-AC-02-76 ERO 3068.

1. Introduction.

In a supersymmetric field theory in Minkowski space, there is a striking relation between the vacuum energy and preservation or spontaneous breakdown of supersymmetry [1,2]. Exact supersymmetry implies that the vacuum energy vanishes to all finite orders of perturbation theory, and conversely. Spontaneous breakdown implies that the vacuum energy is positive, and conversely. Heuristically, these facts follow from the positivity of the energy operator $H \sim \sum_\alpha \{Q_\alpha, Q_\alpha^+\}$ where Q_α is the spinor charge.

In this note we explore the corresponding situation in supersymmetric field theories in anti-de Sitter space which occurs as a natural space-time background in supergravity theories both in $d = 4$ [3] and higher [4] dimensions. The large cosmological constant is a serious problem in such theories, and one may view the present work as part of a long range study of that problem.

In a fixed AdS background of $d = 2$ or 4 dimensions, supersymmetric field theories are invariant under transformations of the supergroup $OSp(N, d)$. For the simplest case $N = 1$, there is the anticommutator

$$\begin{aligned} \{Q_\alpha, \overline{Q}_\beta\} &= 2(\gamma^a M_{ad} + i\sigma^{ab} M_{ab})_{\alpha\beta} \\ \{\gamma^a, \gamma^b\} &= 2\eta^{ab} \\ \frac{1}{4}[\gamma^a, \gamma^b] &= \sigma^{ab} \end{aligned} \tag{1}$$

which relates the spinor charge Q_α to the generators M_{AB} of the space-time group $O(d-1, 2)$. In our notation the indices A, B,... range from 0 to d, while a, b (and μ, ν) range from 0 to $d-1$.

The energy operator $H = M_{od}$ is formally positive (as is quickly shown by a suitable trace of (1)) and its expectation value in a supersymmetric vacuum would be expected to vanish. Yet two problems suggest that all is not well with the conventional interpretation and motivate the present study.

A) The classical energy $\overline{E} = \int d^{d-1}x\sqrt{-g}T_0^0$ computed by standard procedure does not have a positive definite integrand (for scalar fields) [3] and can have arbitrary sign even in a supersymmetric vacuum.

B) A computation [5] of the self-energy of free supermultiplets by a ζ-function regularization technique yielded non-zero results (for $N < 5$) in $(AdS)_4$, whereas in flat space there is an exact cancellation between boson and fermion contributions.

One recent approach to the paradox raised by B) was the observation [6] that if Q_α is represented in terms of free field creators and annihilators, then the Hamiltonian operator computed from $\frac{1}{2}\{Q_\alpha, Q_\alpha^+\} = H$ naturally appears in normal ordered form and thus annihilates the vacuum. It was concluded that the ζ-function regularization calculation violates supersymmetry. This is also one direct conclusion of the present work, but our approach is rather different from [6]. In a class of two-dimensional supersymmetric theories with interactions we define the spinor charge Q_α as an integral of the supercurrent. We then calculate $\{Q_\alpha, \overline{Q}_\beta\}$ by a method equivalent to canonical commutation relations, and obtain space-time generators M_{AB} which are "improved" with respect to the standard ones by the addition of surface terms. The improved energy $E = \int dx \sqrt{-g}T^0_{\ 0}$ has a non negative integrand which vanishes only in supersymmetric field configurations. We show that the corresponding operator H has vanishing expectation value in a supersymmetric vacuum to all orders in perturbation theory. The situation of vacuum energy in AdS is thus essentially the same as in flat-space supersymmetry. Some differences, related to vacuum stability and boundary conditions, are discussed in Section IX.

II. The Geometry of $(AdS)_2$:

Let us begin the discussion with a quick review of the geometry of $(AdS)_2$ which is the hyperboloid $\eta_{AB}y^A y^B = a^{-2}$ embedded in R^3 with Cartesian coordinates y^A and flat metric $\eta_{AB} = (+-+)$. One can introduce intrinsic coordinates t, ρ via

$$y^0 = a^{-1} \sin t \sec \rho \qquad y^1 = a^{-1} \tan \rho \qquad y^2 = -a^{-1} \cos t \sec \rho \tag{2}$$

and the line element induced from the embedding takes the conformally flat form

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \frac{1}{a^2 \cos^2 \rho}(dt^2 - d\rho^2) \tag{3}$$

The hyperboloid is covered once if one takes $-\pi/2 < \rho < \pi/2$ and $-\pi \leq t < \pi$ but to avoid closed timelike curves and to incorporate fields of arbitrary Lagrangian mass it is conventional to pass to the covering space by removing the restrictions on the range of t.

The non-vanishing Christoffel symbols are $\Gamma^t_{t\rho} = \Gamma^t_{\rho t} = \Gamma^\rho_{tt} = \Gamma^\rho_{\rho\rho} = \tan \rho$. With zweibein $V^a_\mu = (a \cos \rho)^{-1} \delta^a_\mu$ one can compute the spin connection $\omega^{\hat{0}\hat{1}}_t = -\tan \rho$ and $\omega^{\hat{0}\hat{1}}_\rho = 0$. The curvature is $R_{\mu\nu\rho\sigma} = a^2(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\mu\rho})$, and $R_{\mu\nu} = a^2 g_{\mu\nu}$.

To formulate supersymmetric theories one needs the notion of Killing spinors, which are spinors $\epsilon(x)$ which satisfy

$$D_\mu \epsilon(x) = -\frac{1}{2} i a \gamma_\mu \epsilon(x) \tag{4}$$

In the coordinate system and frame used here, the solutions of (4) are $\epsilon(x) = S(x)\xi$ where ξ_α is a constant spinor and

$$S(x) = (\cos\rho)^{-1/2}[\cos\frac{\rho}{2} + i\gamma^{\hat{1}}\sin\frac{\rho}{2}]e^{-i\gamma^{\hat{0}}t/2} \tag{5}$$

which satisfies $\overline{S}(x)S(x) = 1$, with $\overline{S}(x) = \gamma^{\hat{0}}S^+(x)\gamma^{\hat{0}}$, and

$$\overline{S}(x)\gamma^\mu S(x) = \gamma^a K^\mu_{a2} + i\sigma^{ab}K^\mu_{ab} \tag{6}$$

where the K^μ_{AB} are Killing vectors which generate the $O(2,1)$ isometry group of the hyperboloid, specifically $K_{AB} = K^\mu_{AB}\frac{\partial}{\partial x^\mu}$ and

$$\begin{aligned} K_{02} &= \frac{\partial}{\partial t} \\ K_{12} &= -\sin t \sin\rho\frac{\partial}{\partial t} + \cos t\cos\rho\frac{\partial}{\partial\rho} \\ K_{01} &= \cos t\sin\rho\frac{\partial}{\partial t} + \sin t\cos\rho\frac{\partial}{\partial\rho} \end{aligned} \tag{7}$$

The first of these is the time translation generator which is conjugate to energy.

III. The model and its conserved charges and vacua.

We now come to the model which involves a multiplet of real fields $\phi(x)$, $\psi(x)$, $F(x)$ with transformation rules

$$\begin{aligned} \delta\phi(x) &= \bar{\epsilon}(x)\phi(x) \\ \delta\psi(x) &= (-i\,\not{\partial}\phi(x) + F(x))\epsilon(x) \\ \delta F(x) &= -i\bar{\epsilon}(x)\,\not{D}\psi(x) \end{aligned} \tag{8}$$

Using Killing spinors (4) one can show that these generate the algebra (1). For a general superpotential $W(\phi)$, the kinetic and interaction terms of the following action are separately invariant under (8),

$$S = \frac{1}{2}\int d^2x\sqrt{-g}\{\partial^\mu\phi\partial_\mu\phi + i\overline{\psi}\gamma^\mu D_\mu\psi + F^2 + 2FW' + 2aW - W''\overline{\psi}\psi\} \tag{9}$$

All indices are raised by the inverse metric and zweibein of $(AdS)_2$; the spin connection in $D_\mu = \partial_\mu + \frac{1}{2}\omega_{\mu ab}\sigma^{ab}$ is actually optional for Majorana spinors. The model is easily generalized to include arbitrary numbers of multiplets ϕ_i, ψ_i, F_i. Note that the action depends on the superpotential $W(\phi)$ as well as derivatives W' and W'', and that one obtains the conventional form in the flat space limit ($a \mapsto 0$).

The generators of $O(2,1)$ transformations and the supersymmetry charges can be obtained by the method of Noether's theorem, which yields

$$\overline{M}_{AB} = \int d\rho \sqrt{-g}\, T^0_\nu K^\nu_{AB}$$

$$Q_\alpha = \int d\rho \sqrt{-g}\, \overline{S}(x)_{\alpha\beta} J^0_\beta \tag{10}$$

with stress tensor and supercurrent

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi + \frac{i}{4}\overline{\psi}(\gamma_\mu D_\nu + \gamma_\nu D_\mu)\psi - g_{\mu\nu}[\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}(W'(\phi))^2 + aW(\phi)] \tag{11}$$

$$J^\mu = (\not{\partial}\phi + iW'(\phi))\gamma^\mu\psi \tag{12}$$

where the fermion equation of motion has been used and the auxiliary field F has been eliminated (see below). The stress tensor satisfies the expected covariant conservation law $D^\mu T_{\mu\nu} = 0$, but the supercurrent satisfies $D_\mu J^\mu = -\frac{ia}{2}\gamma_\mu J^\mu$ which gives a conserved spinor charge because of (4).

The equations of motion are

$$\frac{\delta S}{\delta F} = F + W'(\phi) = 0$$

$$\frac{\delta S}{\delta \phi} = - \quad \phi + FW''(\phi) - \frac{1}{2}W'''(\phi)\overline{\psi}\psi = 0 \tag{13}$$

$$\frac{\delta S}{\delta \overline{\psi}} = (i\gamma^\mu D_\mu - W''(\phi))\psi = 0$$

Any $O(2,1)$ invariant vacuum state has $<\phi(x)>$ = constant, and $<\overline{\psi}\psi>$ vanishes classically. Further, $< F(x) >= 0$ is the signal of a supersymmetry preserving ground state, while $< F(x) > \neq 0$ indicates supersymmetry breaking.

Thus the possible $O(2,1)$ invariant classical states of the system are critical points of the potential

$$V(\phi) = \frac{1}{2}(W'(\phi))^2 - aW(\phi) \tag{14}$$

i.e.,

$$V'(\phi) = W'(\phi)(W''(\phi) - a) = 0$$

Any root ϕ_s of $W'(\phi) = 0$ is a possible supersymmetric state, while any root ϕ_b of $W''(\phi) = a$ is a possible state of broken symmetry. In the latter case $W''(\phi_b) = a$ is the Lagrangian mass appropriate to a Goldstone spinor in $(AdS)_2$. The non-zero value is related to the fact that the covariant divergence of the supercurrent is not zero. (See [7] for a discussion of the 4-dimensional case.) If neither $W'(\phi) = 0$ nor $W''(\phi) = a$ can be satisfied, then the system has no $O(2,1)$ invariant classical states.

We now note that the scalar terms of the energy density in (11) are not necessarily positive, and for supersymmetric states, the energy density is $T_0^0 = V(\phi_s) = -aW(\phi_s)$ which is of arbitrary sign. This is the situation noted in the introduction.

We still do not know whether the states discussed above are stable. To study this we examine M_{02} for small fluctuations $h(x) = \phi(x) - \phi$ with Lagrangian mass

$$V''(\phi) = W''(\phi)(W''(\phi) - a) + W'(\phi)W'''(\phi) \tag{15}$$

which can be of any sign. If $V''(\phi) < 0$ we perform a scaling argument similar to [3]. After scaling $h(x) = (\cos\rho)^\lambda \tilde{h}(x)$ and integration by parts we find that the total energy is convergent and positive for real $\lambda = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{V''(\phi)}{a^2}}$ so that $V''(\alpha) > -\frac{1}{4}a^2$ is required for stability. One also finds that $h(x) \sim (\cos\rho)^\lambda$ is the falloff condition of regular solutions of the fluctuation wave equation $\quad h + V''(\phi)h = 0$, and that for $V''(\phi) > -\frac{1}{4}a^2$ the fluctuation wave functions are the basis of a unitary irreducible representation of $O(2,1)$. For supersymmetric critical points, $W'(\phi_s) = 0$, the stability condition is always satisfied. For non-supersymmetric critical points the condition becomes $W'(\phi_b)W'''(\phi_b) > -\frac{1}{4}a^2$ which may or may not be satisfied.

V. Improved $O(2,1)$ Generators

Let us now study the energy functional

$$\overline{M}_{02} = \frac{1}{2}\int d\rho \left\{(\partial_t\phi)^2 + i\overline{\psi}\gamma_0\partial_t\psi + \frac{(W'(\phi))^2 - 2aW(\phi)}{a^2\cos^2\rho}\right\} \tag{16}$$

The scalar part is of indefinite sign due to the term $-2aW(\phi)$. However, we can take this term, partially integrate using $\partial_\rho \tan\rho = \cos^{-2}\rho$, complete the square in $\partial_\rho\phi$ and find a new energy operator

$$M_{02} = \frac{1}{2}\int d\rho\{(\partial_t\phi)^2 + (\partial_\rho\phi + a^{-1}\tan\rho W'(\phi))^2 + i\overline{\psi}\gamma_0\partial_t\psi + a^{-2}W'(\phi)^2\} \tag{17}$$

which differs from the first by a surface term

$$M_{02} = \overline{M}_{02} + \lim_{\rho\mapsto\pi/2} a^{-1}\tan\rho[W(\phi(t,\rho)) + W(\phi(t,-\rho)] \tag{18}$$

Classically, the new expansion has the desired property that the scalar terms in the integrand are positive semi-definite and vanish only in supersymmetric backgrounds where $W'(\phi_s) = 0$. In such a background we expect to deal with field configurations of the form $\phi(x) = \phi_s + h(x)$ where $h(x)$ is a fluctuation field which can be large in the interior, but decays as $\rho \mapsto \pm\pi/2$ at the rate $h(x) \sim (\cos\rho)^\lambda$ of regular solutions of the linearized wave equation, i.e., $\lambda = \frac{1}{2} + |a^{-1}W'' - \frac{1}{2}|$. For such field configurations the surface term is an irrelevant infinite constant, independent of the fluctuation $h(x)$.

In this paper we implicitly consider only the regular solutions of the linearized wave equations. However, it appears that the irregular solutions $h(x) \sim (\cos\rho)^{\overline{\lambda}}$ with $\overline{\lambda} = \frac{1}{2} - |a^{-1}W'' - \frac{1}{2}|$ will be necessary when $0 < a^{-1}W'' < \frac{1}{2}$. A detailed analysis of free-field theory in $(AdS)_2$ and its relation to stability and unitary representations of $O(2,1)$ should still be performed. The analysis would be similar to the $(AdS)_4$ analysis of [3]. However, we expect that the awkward improvement procedure there will be supplanted by the present method, and the energy defined by (17) will be valid for both regular and irregular modes.

We now wish to argue that M_{02} is the "right" energy operator in the quantum theory, and we wish to proceed in a way which is independent of the coordinatization of the $(AdS)_2$ background and uniform for all generators M_{AB} of $O(2,1)$. Therefore, we calculate $\{Q_\alpha, \overline{Q}_\beta\}$ to determine the specific form of the $O(2,1)$ generators which appear in (1). For this purpose we compute the change of the supercurrent J^μ of (12) under a supersymmetry transformation with Killing spinor parameter $\epsilon(x) = S(x)\xi$. Specifically, we compute

$$\delta_\epsilon \bar{\epsilon}' J^\mu = i[\bar{\xi} Q, \bar{\epsilon}' J^\mu] \tag{19}$$

using (8) and (12). If we write Q_α in more covariant form by integrating over a spacelike surface Σ with normal surface element $d\Sigma_\mu = \epsilon_{\mu\nu} dx^\nu$ (with $\epsilon_{\mu\nu}$ a tensor, i.e., $\epsilon_{01} = \sqrt{-g}$), then

$$\begin{aligned}\delta_\epsilon \int d\Sigma_\mu \bar{\epsilon}' J^\mu &= i[\bar{\xi} Q, \bar{\xi}' Q] \\ &= i\bar{\xi}_\alpha \{Q_\alpha, \overline{Q}_\beta\} \xi'_\beta\end{aligned} \tag{20}$$

The computation of $\delta_\epsilon \bar{\epsilon}' J^\mu$ involves some fancy Fierz-work, repeated use of (4), and the identity $D^\nu D_\nu K^\mu = -a^2 K^\mu$ for Killing vectors of $(AdS)_2$. Total derivative boundary terms are not discarded. The result is

$$\delta_\epsilon \int d\Sigma_\mu \bar{\epsilon}' J^\mu = 2i \int d\Sigma_\mu \{\bar{\epsilon}\gamma_\nu \epsilon' T^{\mu\nu} - a^{-1} D_\nu (W D^\nu (\bar{\epsilon}\gamma^\mu \epsilon'))\} \tag{21}$$

Since $\bar{\epsilon}\gamma^\mu\epsilon'$ is a linear combination of $O(2,1)$ Killing vectors, see (6), with the matrices γ^a, and $i\sigma^{ab}$ which occur in (1), we can identify the $O(2,1)$ generators of the superalgebra, as

$$M_{AB} = \int d\Sigma_\mu \{K_{AB\nu} T^{\mu\nu} - a^{-1} D_\nu (W D^\nu K^\mu_{AB})\} \tag{22}$$

Thus we see that the $O(2,1)$ generators M_{AB} of the superalgebra differ from those constructed by Noether's theorem (10) by the second term in (22) which is a covariant boundary term (but which is kept as a contribution to the integrand of M_{AB} in all of the following).

It is well known that the canonical stress tensor $T^{\mu\nu}$ can be improved by adding the quantity

$$\Delta T^{\mu\nu} = (g^{\mu\nu}\Box - D^\mu D^\nu + R^{\mu\nu})S \tag{23}$$

which is identically conserved in $(AdS)_d$ for any scalar function S of the fields, and that this contributes only a boundary term to the $O(d-1,2)$ charges [3]. However, the boundary term in (22) can only be written in the form (23), with $S \sim W(\phi)$, plus an extra boundary term, proportional to $D_\nu(K^{[\mu}_{AB} D^{\nu]} W)$ which is of superpotential form and thus identically conserved. The supercurrent can also be improved by adding

$$\Delta J^\mu = \sigma^{\mu\nu}(D_\nu - i\frac{a}{2}\gamma_\nu) S\psi \tag{24}$$

which identically satisfies the relevant "conservation" equation for any scalar S, and adds a boundary term to the supercharge Q_α. The supercurrent improvement would change the generators M_{AB} by further boundary terms which we have not computed in detail, since physicallly satisfactory M_{AB} have been obtained without such improvement.

Let us return to (22) and note that the energy operator M_{02} becomes identical with (17) when evaluated in the previous coordinate system. The scalar contribution to the energy density therefore has the desired positivity properties. It may be possible to prove positivity in a fully covariant manner, i.e, for all spacelike surfaces and all globally positive timelike Killing vectors, but for the present we will be content with the previous coordinate dependent proof.

V. $< M_{AB} >$ Vanishes!

The next step is to show that the vacuum expectation values $< M_{AB} >$ of the operators (22) vanish in a supersymmetric vacuum. We will do this covariantly to all orders in perturbation theory, but in a formal manner which does not take the necessary care with ultraviolet divergences. In the next section we will perform a more careful regulated calculation to one-loop order with the same vanishing result. The formal argument requires some theoretical development which we now begin.

By applying (8) repeatedly we easily derive the transformation properties of the superpotential operator $W(\phi(x))$

$$\delta W = \bar{\epsilon} W' \psi$$

$$\delta W'\psi = [-i \not{\partial} W + W'F - \frac{1}{2} W''\overline{\psi}\psi]\epsilon \tag{25}$$

$$\delta(W'F - \frac{1}{2} W''\overline{\psi}\psi) = -i\bar{\epsilon}(\not{\partial} W'\psi + W' \not{D}\psi)$$

which shows that W, $W'(\phi)\psi$ and $W'F - \frac{1}{2}W''\psi\psi$ transform as the ϕ, ψ and F components of a composite multiplet. Using (4), the second expression in (25) can be conjugated and rewritten as

$$[i \not{\partial} W + W'F - \frac{1}{2} W''\overline{\psi}\psi]_{\alpha\beta} = iS_{\alpha\gamma}(x)\{Q_\gamma, W'(\phi)\overline{\psi}_\beta\} \tag{26}$$

We now take the vacuum expectation value of (26), and use the fact that $\partial_\mu < W(\phi(x)) > = 0$ as a consequence of $O(2,1)$ invariance, and that the expectation value of the right side vanishes if the vacuum is supersymmetric, which we assume. Finally, we use the operator field equation

$F(x) = -W'(\phi(x))$, which is correct because the transformation rules (8) are compatible with this equation of motion. (Indeed, we could have used $F = -W'$ initially in (25) and (26).) Thus we have proved that

$$< W'(\phi)^2 + \frac{1}{2}W''(\phi)\overline{\phi}\phi(x) >= 0 \tag{27}$$

Let us now use (27) to prove that $< M_{AB} >= 0$. First note that $< T^{\mu\nu} >= \frac{1}{2}g^{\mu\nu} < T^{\rho}_{\rho} >$ in $(AdS)_2$, because the only $O(2,1)$ invariant, conserved symmetric tensor is the metric $g^{\mu\nu}$. Evaluating the trace from (11), and using the spinor field equation in (13), we find

$$\begin{aligned} < T^{\rho}_{\rho} > &=< W'^2 + \frac{1}{2}W''\overline{\psi}\psi - 2aW > \\ &= -2a < W(\phi) > \end{aligned} \tag{28}$$

where (27) is used in the last step. The expectation value of (22) is thus

$$\begin{aligned} < M_{AB} > &= \int d\Sigma_\mu \{ K_{AB\nu} < T^{\mu\nu} > -a^{-1} < W > D_\nu D^\nu K^\mu_{AB} \} \\ &= \int d\Sigma_\mu \{ -aK^\mu_{AB} < W > +aK^\mu_{AB} < W > \} \\ &= 0 \end{aligned} \tag{29}$$

where we have used again $D^\nu D_\nu K^\mu = -a^2 K^\mu$. Thus $< M_{AB} >= 0$ in a supersymmetric vacuum because of an exact cancellation in the integrand between the contribution of $< T^{\mu\nu} >$ and the contribution of $< W >$. Since in general $< W > \neq 0$, and W determines the boundary terms by which M_{AB} and $\overline{M}_{AB}$ differ, we see that there is no reason to expect that the naive generators $\overline{M}_{AB}$ have vanishing vacuum expectation value.

Although the previous argument used formal operator manipulations without regard for ultraviolet renormalization, we believe that it is entirely correct for the following reason. The action (9) can be regularized by adding Pauli-Villars supermultiplets with suitably chosen couplings to the physical fields. (See [8] for a discussion in flat superspace.) One would then have a regularized $T_{\mu\nu}$ and W with ultraviolet finite vacuum expectation values. Eq. (27) would hold (in a form generalized to include couplings to regulator fields) and the formal trace manipulation which leads to (28) would be valid. Thus $< M_{AB} >= 0$ would be correct in the regularized theory, and, by continuity, would hold in the limit of large regulator masses.

VI. One loop computation of $< M_{AB} >$

It is this idea which we will now implement in a one-loop calculation of $< M_{AB} >$. We will first need some information to set up perturbation theory in $(AdS)_2$, and for this purpose we look at the action of the free massive supermultiplet

$$S = \frac{1}{2}\int d^2\alpha\sqrt{-g}\{\partial^\mu\phi\partial_\nu\phi + i\bar{\psi}\gamma^\mu D_\mu\psi - (\mu^2 - a\mu)\phi^2 - \mu\bar{\psi}\psi\} \tag{30}$$

obtained by setting $W = \frac{1}{2}\mu\phi^2$ in (9) and eliminating F.

The Feynman propagator $G_F(x,x') = -i < T\phi(x)\phi(x') >$ satisfies

$$(\Box_x + m^2)G_F(x,x') = -(-g)^{-\frac{1}{2}}\delta(x,x') \tag{31}$$

As in all maximally symmetric spaces, $O(2,1)$ invariance implies that the propagator is a function of the variable $\eta_{AB}y^Ay'^B$ where y^A (and y'^B) are related to x^μ (and x'^μ) as in (2). It is most convenient to define the variable

$$u = \frac{1}{2}a^2(y^A - y'^A)^2$$

$$= \frac{1}{2}[1 - \sec\rho\sec\rho'(\cos(t-t') - \sin\rho\sin\rho')] \tag{32}$$

which is one-half the chordal distance between the points y^A and y'^B on the hyperboloid of unit scale $a = 1$. The left side of (31) becomes the hypergeometric equation. One chooses the solution which has the same behaviour at spatial infinity as regular solutions of the free-field equation $(\Box + m^2)\phi(x) = 0$, and one normalizes to reproduce the standard short distance singularity of two-dimensional field theory. The result is

$$G(x,x') = \frac{-i}{4\pi}\frac{\Gamma^2(\lambda)}{\Gamma(2\lambda)}(-u)^{-\lambda}F(\lambda,\lambda;2\lambda;u^{-1})$$

$$\approx_{u\to 0} \frac{-i}{4\pi}[-\log(-u) + 2\psi(1) - 2\psi(\lambda) + m^2u\log(-u) + O(u^2)] \tag{33}$$

$$\lambda = \frac{1}{2} + \sqrt{\frac{1}{2} + \frac{m^2}{a^2}}$$

where $\psi(z) = \frac{d}{dz}\log\Gamma(z)$. The standard analyticity property of $F(a,b;c;z)$ corresponds to the time ordered Green's function, although the strict notion of time ordering must be modified [9] because of the peculiar causal properties of AdS. The asymptotic formula in (33) was obtained from a standard mathematical reference [10]. The scalar propagator in $(AdS)_4$ has been obtained from similar considerations [11]. The scalar propagator (33) is actually a Legendre function $Q_{\lambda-1}(1-2u)$, [10].

We also need the spinor propagator. This is easily obtained from the scalar propagator by a Ward-Takahaishi identity. This identity is obtained by the standard method of inserting S of [30] in a path integral generating functional with sources and performing a supersymmetry transformation (8). It is convenient to write the result as

$$\overline{S}(x) < T\psi(x)\overline{\psi}(x') > S(x') = (\overline{S}(x)i\gamma^{\mu}S(x)\partial_{\mu} + \mu) < T\phi(x)\phi(x') >$$

$$= (ia\gamma^{a}K_{a2} - a\sigma^{ab}K_{ab} + \mu) < T\phi(x)\phi(x') > \tag{34}$$

where differentiation is with respect to x, and (6) has been used. Note that (34) relates the spinor propagator for mass μ to the scalar propagator for mass $m^2 = \mu^2 - a\mu$. Note that (34) correctly defines the propagator for a fermi field of mass μ in $(AdS)_2$, whether or not the theory in which it appears is supersymmetric. One can verify directly from (34) that $(i\not{D} - \mu) < T\psi(x)\overline{\psi}(x') >= -i(-g)^{-\frac{1}{2}}\delta(x,x')$.

To calculate $< T^{\mu\nu} >$ using Pauli-Villars regularization we take a set of multiplets ϕ_i, ψ_i with mass parameters μ_i. We then define

$$< T_{\mu\nu} >_{reg} = \sum_i c_i < \partial_{\mu}\phi_i\partial_{\nu}\phi_i + \frac{i}{2}\overline{\psi}\gamma_{\mu}D_{\nu}\psi - g_{\mu\nu}[\frac{1}{2}(\partial\phi_i)^2 - \frac{1}{2}(\mu_i^2 - a\mu_i)\phi_i^2] > \tag{35}$$

where the physical multiplet ϕ, ψ appears as ϕ_0, ψ_0 in the sum with $c_0 = 1$ and $\mu_0 = \mu$. On dimensional grounds $< T_{\mu\nu} >_{reg}$ will be finite provided that the sum rules $\sum_i c_i = \sum_i c_i\mu_i = \sum_i c_i\mu_i^2 = 0$ hold. Then we can take the trace and obtain

$$< T^{\mu}_{\mu} >_{reg} = \sum c_i[(\mu_i^2 - a\mu_i) < \phi_i^2 > + \frac{1}{2}\mu_i < \overline{\psi}_i\psi_i >]$$

$$= \sum c_i[(\mu_i^2 - a\mu_i) < \phi_i^2 > - \mu_i^2 < \phi_i^2 >] \tag{36}$$

where the regularized form of (34) has been used to express $< \overline{\psi}\psi >$ in terms of $< \phi^2 >$. We see that

$$< T^{\mu}_{\mu} >_{reg} = -a\sum c_i\mu_i < \phi_i^2 > \tag{37}$$

However, by the same method one can calculate

$$< W(\phi) >_{reg} = \frac{1}{2}\sum_i c_i\mu_i < \phi_i^2 > \tag{38}$$

which is a finite quantity. Thus we find that (28) holds in 1-loop order for all values of the regulator mass. This immediately leads to the desired result, $< M_{AB} >= 0$.

We have not studied the limit of infinite regulator mass explicitly, but we anticipate that the divergence which appears there can be cancelled by adding an infinite constant counter term to the superpotential. One should note that there is no trace anomaly in this theory. Although $T^{\mu}_{\mu} = \alpha R$ for two-dimensional free field theory in a background metric with scalar curvature R, the real scalar and Majorana spinor fields contribute to the coefficient α with equal and opposite sign.

VII. ς-Function Calculation of the Vacuum Energy

The 1-loop calculation of $< M_{02} >$ by means of supersymmetric Pauli-Villars regularization of $< T^{\mu\nu} >$ and $< W >$ in the integrand of (22) can be compared with a calculation of the vacuum energy by ς-function regularization. Since such calculations have been performed [5,13] only for $(AdS)_4$, we must now perform the calculation in $(AdS)_2$ in order to compare results directly. The basic idea is to sum the energy eigenvalues ω_i of states of a unitary positive-energy irreducible representation of $O(2,1)$ and to regulate this divergent sum using the ς-function.

The relevant representations of $O(2,1)$, or, more precisely, its universal covering group, are denoted by a real number $\omega_0 > \frac{1}{2}$ which is the lowest eigenvalue of $a^{-1}M_{02}$. The energy eigenvalues are then spaced by integer with respect to ω_0, i.e., we have $\omega_n = (\omega_0 + n)$ and each level $n = 0, 1, 2, \ldots$ has unit multiplicity. For the free supersymmetric field theory (30), with $\frac{\mu}{a} > \frac{1}{2}$, the scalar field corresponds to a representation with $\omega_0 = \frac{\mu}{a}$, and the fermi field to a representation with $\omega_0 = (\frac{\mu}{a}) + \frac{1}{2}$. These representations combine to form a unitary irreducible representation of the superalgebra $OSp(1,2)$. For $(AdS)_4$ the correspondence between free fields and representations of $SO(3,2)$ and $OSp(1,4)$ has been discussed previously [14,15,3].

The regulated self energies of the boson (B) and the fermion (F) are defined by

$$E_B(\omega_0, z) = a \sum_{n=0}^{\infty} (\omega_0 + n)^{-z} = a\varsigma_R(z, \omega_0)$$

$$E_F(\omega_0, z) = a \sum_{n=0}^{\infty} (\omega_0 + \frac{1}{2} + n)^{-z} = a\varsigma_R(z, \omega_0 + \frac{1}{2}) \tag{39}$$

and are related to the extended Riemann ς-function ς_R as indicated. The physical values are defined by analytic continuation to $z = -1$, where ς_R can be expressed as a Bernoulli polynomial [16] and we have

$$E_B(\omega_0, -1) = \frac{1}{2}a(-\omega_0^2 + \omega_0 - \frac{1}{6})$$

$$E_F(\omega_0 + \frac{1}{2}, -1) = \frac{1}{2}a(-(\omega_0 + \frac{1}{2})^2 + (\omega_0 + \frac{1}{2}) - \frac{1}{6}) \tag{40}$$

The total self-energy is the difference between these two and we have

$$E_B(\omega_0, -1) = \frac{1}{2}a(\omega_0 - \frac{1}{4}) \tag{41}$$

Thus the energy ς-function method predicts non-zero vacuum energy in a supersymmetric vacuum. We conclude that this method of regularization breaks supersymmetry. For the massless supermultiplets [5] of gauged extended supergravity for $d = 4$ and $N \geq 5$ and for the short representations [13] of $OSp(8,4)$ which occur in the round S_7 Kaluza-Klein solution of $D = 11$ supergravity, the ς-function method does give zero vacuum energy. We believe that this indicates that the vacuum energy is less singular in these particular field theories and therefore less sensitive to the method of calculation. Calculations of the $SO(N)$ coupling constant renormalization [16] support this interpretation.

VIII. On the partition function:

Let us note that an argument very similar to that of Section V was used in [1] to show that the value of the partition function Z is independent of the Yukawa coupling constant for the Wess-Zumino model in $d = 4$ Minkowski space. In flat space $\log Z = -i \int d^4x\, \mathcal{E}$, where $\mathcal{E}$ is the vacuum energy density. Since $\mathcal{E}$ is independent of the Yukawa coupling, it has the same value in the interacting theory ($g \neq 0$) as in the free theory ($g = 0$). In the free theory $\mathcal{E}$ vanishes since there is an explicit cancellation between boson and fermion mode energies..

In AdS there is no simple relation between the partition function Z and the vacuum energy density $\mathcal{E}$. They are simply different quantities. In Section V we showed that $\mathcal{E} = 0$ in supersymmetric vacua. Now we study the partition function and show that it depends on the interaction in AdS (although interaction independence is recovered in the flat limit $a \mapsto 0$). As one application of this general result we show that the partition function of the free massive supersymmetric theory (30) depends on the mass μ. This fact gives some information about the ratio of determinants of the boson and fermion wave operators in anti-de-Sitter space.

Let us define the partition function

$$Z = \int d\Phi e^{iS} \tag{42}$$

where S is the action (9) and the measure is $d\Phi = \Pi_{x,\alpha} d\phi(x) d\psi_\alpha(x) dF(x)$. Under a small variation δW of the superpotential, Z changes by

$$\delta Z = i \int d\Phi e^{iS} \int d^2x\sqrt{-g}\{i \not{\partial}\delta W + F\delta W(\phi) - \frac{1}{2}\delta W''(\phi)\overline{\psi}\psi + a\delta W(\phi)\} \tag{43}$$

where the first terms can be added to the variational derivative because it vanishes in an $O(2,1)$ invariant state. The sum of the first three terms can be recognized as the transform of the ψ component of a composite supermultiplet as in (25) with W replaced by δW. Eq. (26) then implies that this sum vanishes in a supersymmetric state, so that our basic formula reduces to

$$\delta Z = ia \int d\Phi e^{iS} \int d^2x\sqrt{-g}\delta W(\phi(x)) \tag{44}$$

Let us apply this formula to the free theory (30) with $W = \frac{1}{2}\mu\phi^2$, and obtain

$$\frac{\partial Z}{\partial \mu} = \frac{1}{2} ia \int d\Phi e^{iS} \int d^2x\sqrt{-g}\phi^2(x) \tag{45}$$

This indicates that boson and fermion determinants satisfy

$$\frac{\partial}{\partial \mu}\log\left[\frac{det(\quad + \mu^2 - a\mu)}{det(i \not{D} - \mu)}\right] = ia \int d^2x\sqrt{-g} < \phi^2(x) > \tag{46}$$

Thus the ratio of determinants in $(AdS)_2$ is both ultraviolet and volume divergent.

IX. Summary

What we have done in this paper is to show that the $O(2,1)$ generators which appear in the supersymmetry anticommutator (1) are given by (22). They differ by a covariant surface term from the naive generators (10). The improved energy operator has the properties which we would have expected from flat-space supersymmetry. The classical energy density is positive semi definite and vanishes only in supersymmetric configurations. Quantum mechanically the improved vacuum energy vanishes to all orders if supersymmetry holds.

Nevertheless, there are still some differences between flat space and AdS supersymmetry. In flat space if there is a supersymmetric minimum of the potential, then any supersymmetry breaking stationary point has higher energy and is unstable. In AdS the $O(2,1)$ invariant states correspond to stationary points of the scalar potential $V(\phi) = \frac{1}{2}(W'(\phi))^2 - aW(\phi)$. For some potentials there are supersymmetry breaking states which lie above supersymmetric ones. In other potentials the breaking states lie below the supersymmetric ones. The relative ordering of states in the potential $V(\phi)$ (and the ordering of constant configurations in the improved energy density (17)) is at best an incomplete guide to their relative stability. Supersymmetric states are always stable, since the energy of fluctuations about them is globally positive. Non-supersymmetric states may or may not be stable depending on details of the potential. The basic reason this situation is different from that in flat space has little to do with supersymmetry. It is the fact that penetration of a potential barrier does not always occur in AdS for geometrical reasons [18]. Bubble formation is not always energetically possible. One consequence of this is that a model of the type (9) can actually have several stable ground states, some supersymmetric and some not.

The present results have been established only for $OSp(1,2)$ supersymmetry in $D = 2$. The extension of these ideas to $OSp(1,4)$ supersymmetry in four dimensions is currently in progress.

X. Acknowledgements.

This research was performed at the Aspen Center for Physics, whose hospitality is gratefully acknowledged. We thank our colleagues there, particularly P. Candelas, S. Coleman, H. Nicolai and B. Zumino for useful conversations. One of us (D.Z.F.) thanks G.W. Gibbons for collaboration on the vacuum energy problem in $OSp(1,4)$ supersymmetry. Some of the ideas which arose there were used in the present work.

XI. References

[1] B. Zumino, Nucl. Phys. B89 (1975) 535.

[2] E. Witten, Nucl. Phys. B185 (1981) 513.

[3] P. Breitenlohner and D.Z. Freedman, Phys. Lett. B 115 (1982) 197; Ann. of Phys. 144 (1982) 249.

[4] P.G.O. Freund and M.A.Rubin, Phys. Lett. 97B (1980) 265.

[5] B. Allen and S. Davis, Phys. Lett. 124 B (1983) 353.

[6] N. Sakai and Y. Tanii, Tokyo Institute of Technology preprint T17/HEP-80, June 1984.

[7] B. Zumino, Nucl. Phys. B127, (1977) 189.

[8] S.J. Gates, Jr., M.T. Grisaru, M. Rocek and W. Siegel, "Superspace: One Thousand and One Lessons in Supersymmetry," Benjamin-Cummings Pub. Co., Reading, Mass. 1983, 21.95.

[9] L. Castell, Nucl. Phys. B5 (1968) 601; Nuovo Cimento 61A (1969) 505.

[10] A. Erdelyi, Editor, Higher Transcendental Functions, Vol. I, McGraw Hill, New York 1953.

[11] S.J. Avis, C.J. Isham and D. Storey, Phys. Rev. D18 (1978) 173.

[12] N.D. Birrell and P.C.W. Davies, Quantum Fields in Curved Space, Cambridge University Press, 1982.

[13] G.W. Gibbons and H. Nicolai, CERN preprint TH.3815, 1984.

[14] C. Fronsdal, Phys. Rev. D12 (1975) 3819.

[15] W. Heidenreich, Phys. Lett. 110B (1982) 461.

[16] W. Magnus, F. Oberhettinger and R.P. Soni, "Formulae and Theorems for the Special Functions of Mathematical Physics," Springer Verlag.

[17] S.M. Christensen, M.J. Duff, G.W. Gibbon, and M. Rocek, Phys. Rev. Lett. 45 (1980) 161; T. Curtright, Phys. Lett. 1248 (1983) 353.

[18] S. Coleman and D. DeLuccia, Phys. Rev. D21 (1980) 3305.